Pro-Cyclicality Beyond Business Cycles: The Case of Risk Measures

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Motivation

- Since the introduction of risk-based solvency regulation, accepted idea: **risk measurements** made with 'regulatory' risk measures, are **pro-cyclical**
  - ↗ in times of crisis, they overestimate the future risk
  - ↘ they underestimate it in quiet times
- As a result, all financial institutions (insurance companies, banks, regulatory bodies, ...) have to provide substantial capital in the aftermath of a financial crisis, but far less capital prior to such crises.

  Quoting Gilles Moec, Chief Economist of AXA (Le Monde, 2020/01/21. Translated): *'The major mistake made in 2010 was imposing austerity at the worst moment'*
- Hence our two questions:
  1. How to quantify procyclicality (in the way financial institutions measure risk) ?
  2. How to explain it?

A statistical approach

- The regulatory authorities of various sectors, like the BIS, EIOPA or ESMA, have proposed various solutions to reduce procyclicality (in the context of Basel III, Solvency II, EMIR). Approaches based on a macroeconomic perspective on procyclicality
- Our approach is statistical: we examine (empirically and mathematically) if the way of estimating capital requirements using risk measures (such as VaR and ES) is a possible source of procyclicality.

A 3+ years story (Dec 2016 - Feb. 2020), with ...

Casting:

- Starring role: Marcel Bräutigam (ESSEC CREAR-LPSM Sorbonne Univ.) Doctoral thesis (online): *Pro-cyclicality of Risk Measurements - Empirical Quantification and Theoretical Confirmation*
  (Awarded last month one of the two special mentions at the *Prix des Sciences du Risque 2020*)
- Michel Dacorogna (PrimeRe Sol., Switzerland) and Marie Kratz
  See also the overview 'understanding pro-cyclicality' given online:
  `http://crear.essec.edu/home/crear-in-press`

Two main goals

A - Quantifying the pro-cyclicality:

1. generalize in a simple way the static 'regulatory' risk measure VaR (estimated on past data) to a **dynamic one**
2. test the relevance and the **predictive power** of the SQP risk measure
3. **quantify empirically pro-cyclicality**

For this:

1. Consider the measurement itself as a stochastic process, introducing **Sample Quantile Process** (SQP) as a risk measure
2. (a) Play with the random measure defining the SQP

   (b) Define a **look-forward ratio** to see how the historical estimate of the SQP predicts the risk according to the volatility state
3. Use the **realized volatility** as a marker for the **market state**, ...
   ... analyzing the look-forward **SQP ratio conditioned** to the realized volatility

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B - Looking for explanations, we show that pro-cyclicality may be explained by two factors:

1. the very **way risk is measured**
2. **the clustering and return-to-the-mean of volatility**

For this:

1. Consider a simple **iid model** to show

   ... a negative correlation between the logarithm of the SQP ratio and the volatility

   ... empirically and theoretically

2. Use a simple **GARCH(1,1) model**

   ... to observe similar negative correlation

   ... whatever the fatness of the innovation tail

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Risk Measure: VaR

- In financial markets, most popular risk measure: **Value-at-Risk** (VaR)
- Given a loss random variable L (with cdf F_L), level $\alpha \in (0,1)$

$$\mathrm{VaR}(\alpha) = \inf\{\mathrm{x} \in \mathbb{R} : \mathbb{P}(\mathrm{L} \leq \mathrm{x}) \geq \alpha\} \underset{\text{strict.}\nearrow}{\overset{\mathrm{F_L}\text{cont.}}{=}} \mathrm{F_L^{-1}}(\alpha)$$

- Practically, VaR is estimated as an **empirical quantile**:
  Given a sample of n **historical losses** $(L_1, \cdots, L_n)$, $\alpha \in (0,1)$,

$$\widehat{VaR}_\alpha(L) = F_{n;L}^{-1}(\alpha) = \inf\left\{x : \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{(L_i \leq x)} \geq \alpha\right\} = L_{(\lceil n\alpha \rceil)}.$$

  - Known pro-cyclicality of risk estimation

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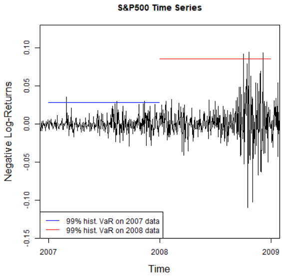

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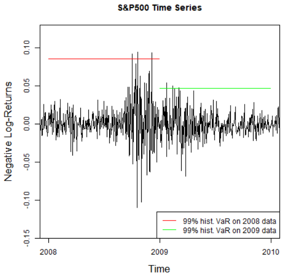


Dynamic extension of VaR: SQP

Consider the measurement itself as a stochastic process:

- **Sample Quantile Process** (SQP) (Miura (92), Akahori (95), Embrechts & Samorodnitsky (95)): Given $L = (L_t, t \geq 0)$, $\alpha \in (0,1)$, a fixed time frame T, and a random measure μ on $\mathbb{R}^+$, the SQP is defined at time t as

$$Q_{T,\alpha,t}(L) = \inf\left\{ x : \frac{1}{\int_{t-T}^{t} \mu(s)ds} \int_{t-T}^{t} \mathbb{1}_{(L_s \leq x)} d\mu(s) \geq \alpha \right\}.$$

- Ex: $\mu=$ Lebesgue measure:
the VaR process $(Q_{T,\alpha,t}(L))_t$ (with a rolling window T)

$$Q_{T,\alpha,t}(L) = \inf\left\{ x : \frac{1}{T} \int_{t-T}^{t} \mathbb{1}_{(L_s \leq x)} ds \geq \alpha \right\}$$

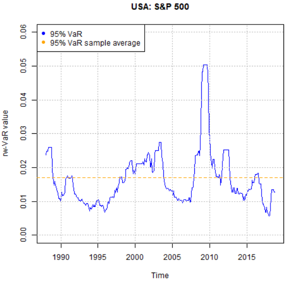
Setup 1: Empirical Study

- Data: **11 stock indices**, daily log-returns from Jan. 1987 to Sept. 2018

- Dynamic 'rolling-window' VaR: $(Q_{T,\alpha,t}(L))_t$ denoted $(\text{VaR}_{T,\alpha,t}(L))_t$, with empirical estimator

$$\widehat{VaR}_{T,\alpha,t}(L) = \inf \left\{ x : \frac{1}{T} \sum_{i \in [t-T,t)} \mathbb{1}_{(L_i \leq x)} \geq \alpha \right\}$$

- For simplicity: $T = 1y$, $\alpha = 95\%$, monthly rolling-window VaR



Setup 2 - Quality of risk prediction

- Introduce a **new quantity**: **look-forward ratio** of VaR's

$$R_{t,\alpha} = \frac{\widehat{VaR}_{1,\alpha,\mathbf{t+1y}}}{\widehat{VaR}_{T,\alpha,t}} \quad \text{with}$$

  $\widehat{VaR}_{T,\alpha,t}$ **used as a predictor** of the risk 1 year later $(t+1y)$
  $\widehat{VaR}_{1,\alpha,\mathbf{t+1y}}$: estimated **realized** risk at time $t+1y$ **(a posteriori)**
  (empirical VaR on 1 year, as asked by regulators)

- $R_{t,\alpha} \approx 1$: correctly assess the 'future risk'
  $R_{t,\alpha} > 1$: under-estimation of the 'future risk'
  $R_{t,\alpha} < 1$: over-estimation of the 'future risk'

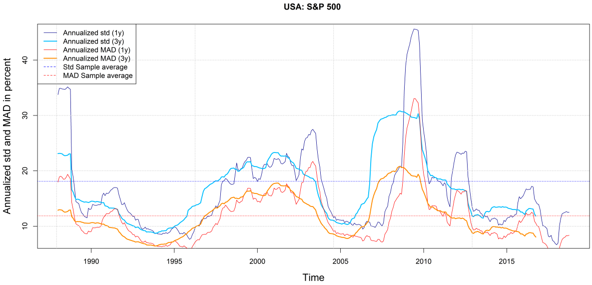
Understanding the Dynamic Behavior

- Use a measure of annualized **realized volatility** as a proxy for **market states**

$$v_{k,n}(t-1) := \sqrt{252} \times \left\{ \frac{1}{n-1} \sum_{i=t-n}^{t-1} \left| X_i - \frac{1}{n} \sum_{i=t-n}^{t-1} X_i \right|^k \right\}^{1/k},$$

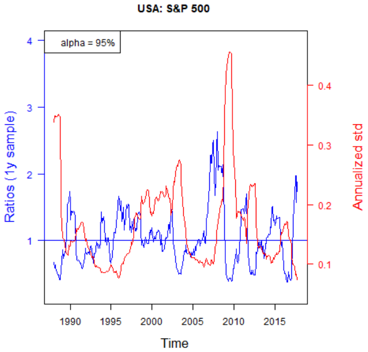
$k=2$: $v_{2,n} = \hat{\sigma}(t)$ empirical standard deviation
$k=1$: $v_{1,n} = \hat{\theta}(t)$ empirical mean absolute deviation (MAD)

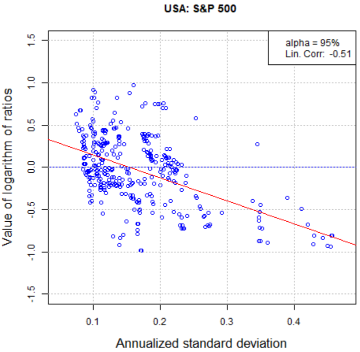
- **Reasonable proxy** to discriminate between quiet and crisis periods



- **Condition** the ratios **on the realized volatility** (indicator of market states: Idea from seminal empirical study on Foreign Exchange Rates by Dacorogna et al. (2001))

Relation between Volatility and look-forward Ratio





- $\log(R_{T,\alpha,t})$ **negatively correlated** with annualized realized volatility:

Volatility year t	SQP log ratio	Meaning
Low Volatility	High Ratio > 0	Underestimation of Risk
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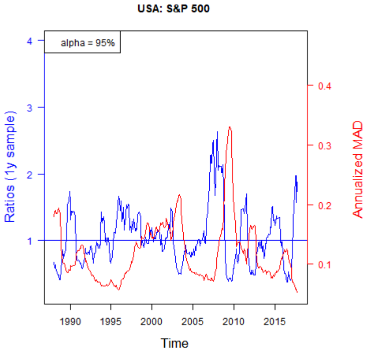

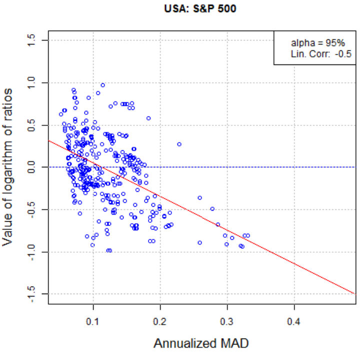

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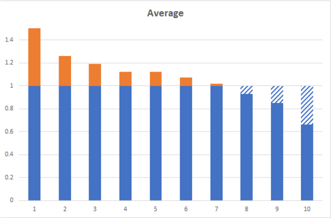
Quantification via volatility binning

An empirical methodology for quantifying procyclicality that can be used for other risk measures (both existing and future measures):

Quantification of the capital requirement as a function of the state of the market



x-axis: states of market volatility (in ascending order from 1 to 10)
y-axis: value of the look-forward ratio (wrt 1)
Under-estimation of capital: shown in orange / Over-estimation : stripped pattern

Two factors that explain the pro-cyclicality

We estimate empirically

$$Cor\left(\log\left|\frac{\widehat{VaR}_{t+1y}}{\widehat{VaR}_t}\right|, \hat{\sigma}_t\right) \text{ and } Cor\left(\log\left|\frac{\widehat{VaR}_{t+1y}}{\widehat{VaR}_t}\right|, \hat{\theta}_t\right)$$

- for an iid model
- for a GARCH(1,1) model
- using different underlying distributions

$\alpha = 95\%$ Model:	Data (average)	GARCH	iid
Correlation (log-ratios) with $\hat{\sigma}_t$	-0.54	-0.63	(-0.19)($t3$)/-0.40 (N)
Correlation (log-ratios) with $\hat{\theta}_t$	-0.51	-0.63	-0.35 ($t3$)/-0.34 (N)

- Part of the pro-cyclicality would be intrinsically due to historic risk estimation ?
- Part would be due to clustering and return to the mean of volatility?

A first factor: the way risk is measured

X parent rv of an iid sample with mean μ, variance σ^2, quantile $q_X(p)$, $p \in (0,1)$

$m(X,r) = \mathbb{E}[|X-\mu|^r]$, $r \geq 1$ ($r=1$: MAD; for $r=2$: std)

THEOREM: ASYMPTOTIC NORMALITY

Under some conditions, the **asymptotic distribution** *of the* **logarithm of the look-forward ratio of the risk measure estimator** *($\log\left|\frac{q_{n,t+1y}(p)}{q_{n,t}(p)}\right|$) with* **the r-th absolute central sample moment** *$\hat{m}(X,n,r)$, is* **bivariate normal**, *with a negative correlation given by $-\frac{1}{\sqrt{2}} \times \frac{|\Gamma_{12}^{(r)}|}{\sqrt{\Gamma_{11}^{(r)}}\sqrt{\Gamma_{22}^{(r)}}}$, where $\Gamma^{(r)}$ is the asymptotic covariance matrix of the bivariate CLT between $q_n(p)$ and $\hat{m}(X,n,r)$.*

Ex: Gaussian case and $r=2$ (std) ($\mathbb{E}[X^4] < \infty$ needed, whereas for $r=1$: $\mathbb{E}[X^2] < \infty$; so think about MAD as a good alternative to std!):

$$\lim_{n\to\infty} \mathrm{Cor}\left(\log\left|\frac{q_{n,t+1y}(p)}{q_{n,t}(p)}\right|, \hat{\sigma}_n\right) = -\frac{1}{\sqrt{2}}\frac{\phi(\Phi^{-1}(p))|\Phi^{-1}(p)|}{\sqrt{2p(1-p)}}$$

A second factor: the clustering and return-to-the mean of volatility

- Use the simplest version of GARCH models, GARCH(1,1), to isolate the effect of clustering of volatility and its return to the mean

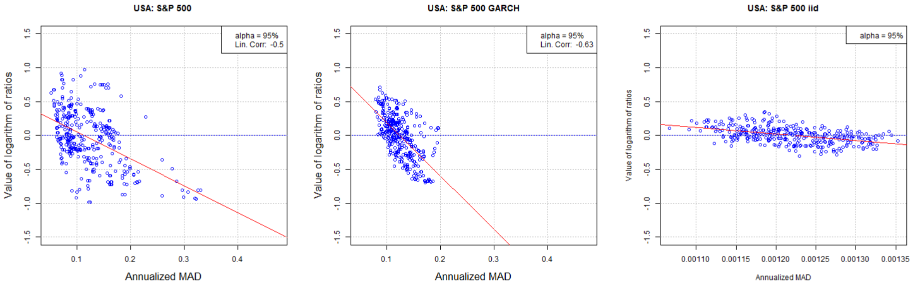
$$\sigma_t^2 = \omega + \alpha\, r_{t-1}^2 + \beta\, \sigma_{t-1}^2 \quad \text{with } r_{t+1} = \sigma_t\, \epsilon_t$$

  where the innovation $\epsilon_t \in \mathcal{N}(0,1)$ or Student, to study the tail effect

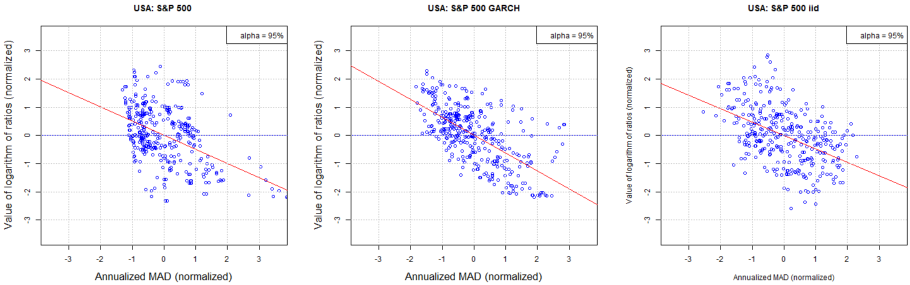
- Fit the parameters ω, α, β to each full sample of the 11 indices, using a robust optimization method (Zumbach 2000) to obtain a stationary solution for the GARCH (s.t. $\boldsymbol{\alpha + \beta < 1}$): the annualized volatility reproduces quite well the realized one (slightly higher)

- Our two estimators, although computed over disjoint samples, might be correlated (in contrast to the iid case). But since the GARCH(1,1) is strongly mixing with geometric rate, it will make the estimators on disjoint samples asymptotically uncorrelated. We also prove in this case the asymptotic bivariate normality of the **log of the look-forward ratio of the risk measure estimator** with the **r-th absolute central sample moment**, with a **negative correlation**

Comparing results



Comparing results



Model:	Data (S&P500)	GARCH	iid (Gaussian)
Correlation (log-ratios)	-0.50	-0.63	-0.34

Conclusion

- Pro-cyclicality of the SQP, a 'dynamic generalization of VaR', confirmed and **quantified** (by conditioning to realized volatility): During high-volatility periods, those risk measures overestimate the risks for the following years, whereas during low-volatility periods, they underestimate them

- Identification of 2 factors explaining pro-cyclicality of risk measurement, with a negative dependence between the realized volatility and the log SQP-ratios shown **empirically** and confirmed **theoretically**
  - (i) **clustering effect of the volatility**, via GARCH models (as expected, but not yet quantified)
  - (i) **the way risk is measured**, via iid model, so independently of business cycles: more surprising and interesting, as it goes beyond economic studies, showing the negative impact that the method of estimating risk can have on risk management)

- Ongoing work: the **design of the SQP** with the proper dynamical behavior as a good basis for **anti-cyclical** regulation